International Journal of Theoretical Physics, Vol. 27, No. 9, 1988

State on Splitting Subspaces and Completeness of Inner Product Spaces

Anatolij Dvurečenskij[1] **and Sylvia Pulmannová**[1]

Received February 28, 1988

We show that an inner product space V is complete iff the system of all splitting subspaces, i.e., of all subspaces M for which $M+M^{\perp}=V$, possesses at least one completely additive state.

1. INTRODUCTION

Let V be a real or complex inner product space with an inner product $(\cdot,\cdot)$. By a subspace of V we shall understand a linear closed submanifold of V.

Denote by $\mathscr{L}(V)=\{M\subseteq V\colon M^{\perp\perp}=M\}$, where $M^{\perp}=\{x\in V\colon (x,y)=0$ for all $y\in M\}$ and by $\mathscr{E}(V)$ the set of all splitting subspaces, i.e., of all M for which $M+M^{\perp}=V$. It is well known that $\mathscr{L}(V)$ is an orthocomplemented complete lattice with the operations $\bigwedge_L$ and $\bigvee_L$ satisfying the equalities

$$\bigwedge_{t\in T}{}_L M_t=\bigcap_{t\in T} M_t,\qquad \bigvee_{t\in T}{}_L M_t=\operatorname{sp}\left(\bigcup_{t\in T} M_t\right)^{\perp\perp} \tag{1}$$

where sp means the linear span. Analogously, $\mathscr{E}(V)$ is an orthocomplemented, orthomodular orthoposet with the operations $\bigwedge_E$, $\bigvee_E$; moreover, $\mathscr{E}(V)\subseteq\mathscr{L}(V)$. The $\mathscr{E}(V)$ contains any complete subspace and therefore any finite-dimensional one. In addition, if $\bigvee_{t}{}_E M_t$ exists in $\mathscr{E}(V)$, then it is equal to $\bigvee_{t}{}_L M_t$.

If V is complete, i.e., V is a Hilbert space, then $\mathscr{E}(V)=\mathscr{L}(V)$ and $\mathscr{E}(V)$ plays a considerable role in the axiomatic model of quantum mechanics: see, for instance, Varadarajan (1968). Hence it is important to find the conditions on $\mathscr{L}(V)$ or $\mathscr{E}(V)$ that characterize Hilbert spaces among inner product spaces.

[1]Mathematical Institute, Slovak Academy of Sciences. CS-814 73 Bratislava, Czechoslovakia.

An elegant characterization is due to Amemiya and Araki (1966): V is complete iff $\mathscr{L}(V)$ is orthomodular. Hence, V is a Hilbert space whenever $\mathscr{L}(V)=\mathscr{E}(V)$.

An interesting measure-theoretic characterization of a separable inner product space has been presented by Hamhalter and Pták (1987): V is complete iff $\mathscr{L}(V)$ possesses at least one state. This result for nonseparable inner product spaces has been generalized due to Dvurečenskij and Mišík (1988).

The completeness characterization through $\mathscr{E}(V)$ has been given by Gross and Keller (1977): V is complete iff $\mathscr{E}(V)$ is a complete lattice; Catteneo and Marini (1986): V is complete iff $\mathscr{E}(V)$ is a σ-lattice and Dvurečenskij (1988b): V is complete iff $\mathscr{E}(V)$ is a quantum logic, i.e., an orthocomplemented, orthomodular, σ-orthoposet.

In the present paper we show that V is complete iff $\mathscr{E}(V)$ possesses a totally additive state. This generalizes the result of Dvurečenskij (1989) from separable inner product spaces to general ones.

2. STATE CRITERION FOR COMPLETENESS

In the present section, we prove the main result of the paper. Let m be a mapping from $\mathscr{E}(V)$ into $[0, 1]$ such that:

1. $m(V)=1$
2. $m\left(\bigvee_{t\in T}^{E} M_t\right)=\sum_{t\in T} m(M_t)$

   whenever $\{M_t: t\in T\}$ is a system of mutually orthogonal splitting subspaces for which the join exists in $\mathscr{E}(V)$.

If 2 holds for any finite index set, any countable or any T, m is said to be a finitely additive state, state, or completely additive state, respectively.

We define these notions in an analogous way for $\mathscr{L}(V)$.

We recall that $\mathscr{E}(V)$ possesses many finitely additive states (Dvurečenskij, 1988b): Let x be a unit vector of V. The mapping m_x: $\mathscr{E}(V)\to[0, 1]$ defined via

$$m_x(M)=\|x_M\|^2, \qquad M\in\mathscr{E}(V)$$

where x_M is a unique vector from M such that $x=x_M+x_{M^\perp}$ $x_{M^\perp}\in M$, is a finitely additive state and $\{m_x: \|x\|=1\}$ is a quite full system of finitely additive states, that is, the statement "if $m_x(M)=1$, then $m_x(N)=1$" implies $M\subseteq N$.

By the dimension of an inner product space we mean the cardinality of any maximal orthonormal set of V. It is clear that any separable inner product space is countable-dimensional, but the converse is not true in general. Indeed, Gudder (1974) for $m = \aleph_0$ and Dvurečenskij (1988a) for general cardinals showed that if m is an infinite cardinal such that $m^{\aleph_0} > m$, then in any Hilbert space H of dimension $m^{\aleph_0}$ there is a dense submanifold V of dimension m containing no orthonormal basis of H.

If x is a nonzero vector from V, then by $\mathrm{sp}(x)$ we mean the one-dimensional subspace of V spanned over x.

Theorem. An inner product space V is complete if and only if $\mathscr{E}(V)$ possesses at least one completely additive state.

Proof. The necessity is simple. Let T be a positive Hermitian operator from V into V of finite trace equal to 1. The mapping m_T defined via

$$m_T(M) = \mathrm{tr}(TP_M), \qquad M \in \mathscr{E}(V) \tag{2}$$

where P_M denotes the orthoprojector from V onto M, is a completely additive state on $\mathscr{E}(V)$. Moreover, Maeda (1980) proved that any completely additive state m on a complete inner product space V, $\dim V \neq 2$, is represented via (2). ■

The sufficient condition follows from the following lemmas.

Lemma 1. Let $\{a_i\}$ and $\{b_j\}$ be maximal orthonormal systems (MONS) in splitting spaces M and $M^\perp$. Then (i) $\{a_i\} \cup \{b_j\}$ is a MONS in V; (ii) $\bigvee_i{}_L\, \mathrm{sp}(a_i) = M$.

Proof. (i) Suppose that $x \in V$ is orthogonal to all a_i and b_j. Express x in the form $x = x_M + x_{M^\perp}$, where $x_M \in M$ and $x_{M^\perp} \in M^\perp$. Then $0 = (x, a_i) = (x_M, a_i) + (x_{M^\perp}, a_i) = (x_M, a_i)$. The maximality of $\{a_i\}$ in M implies $x_M = 0$. Analogously, we prove $x_{M^\perp} = 0$, so that $x = 0$ and $\{a_i\} \cup \{b_j\}$ is an MONS in V.

(ii) Denote

$$M_0 = \bigvee_i{}_L\, \mathrm{sp}(a_i)$$

Then $M_0 \subseteq M$ and $M^\perp \subseteq M_0^\perp$. Let now $x \in M_0^\perp$. Then $x \perp a_i$ for any i. If we put $x = x_M + x_{M^\perp}$, $x_M \in M$, $x_{M^\perp} \in M^\perp$, then $0 = (x, a_i) = (x_M, a_i)$. The maximality of $\{a_i\}$ gives $x_M = 0$. Hence, $x = x_{M^\perp} \in M^\perp$; in other words, $M_0^\perp \subseteq M^\perp$; consequently, $M_0 = M$. ■

Denote by $\mathscr{S}_E(V)$ the set of all completely additive states on $\mathscr{E}(V)$.

Lemma 2. If $\mathcal{S}_E(V) \neq \emptyset$, then, for any unit vector $x \in V$, there is an $s \in \mathcal{S}_E(V)$ such that

$$s(\mathrm{sp}(x)) > 0 \tag{3}$$

Proof. Let m be a completely additive state on $\mathscr{E}(V)$ and choose a MONS $\{x_i\}$ in V. Due to Lemma 1,

$$\bigvee_{i} {}_L \, \mathrm{sp}(x_i) = V$$

The complete additivity of m entails that there exists an x_i with $m(\mathrm{sp}(x_i)) > 0$.

Let x be an arbitrary unit vector of V. Then $M_0 := \mathrm{sp}(\{x, x_i\}) \in \varepsilon(V)$ and we may define a unitary operator $U: V \to V$ such that $Ux_i = x$ and $Uy = y$ for any $y \in M_0^\perp$. The mapping $s: M \mapsto m(U^{-1}(M))$, $M \in \mathscr{E}(V)$, is an element of $\mathcal{S}_E(V)$ with (3). ■

For any $m \in \mathcal{S}_E(V)$ we define the extension $\bar{m}$ of m from $\mathscr{E}(V)$ to $\mathscr{L}(V)$ via

$$\bar{m}(M) = \sup\left\{\sum_i m(\mathrm{sp}(a_i)) \colon \{a_i\} \text{ is a MONS in } M\right\} \tag{4}$$

and put

$$\mathscr{L}_m(V) = \left\{ M \in \mathscr{L}(V) \colon \text{if } \{a_i\} \text{ and } \{b_i\} \text{ are two MONS in } M, \text{ then } \sum_i m(\mathrm{sp}(a_i)) = \sum_i m(\mathrm{sp}(b_i)) \right\}$$

Lemma 1 entails that $\bar{m}(M) = m(M)$ for any $M \in \mathscr{E}(V)$ and $\mathscr{L}_m(V) \supseteq \mathscr{E}(V)$.

Lemma 3. Let $\{x_i\}$ be a nonvoid system of orthomormal vectors from V and put

$$M = \bigvee_{i} {}_L \, \mathrm{sp}(x_i)$$

If $\{y_i\}$ is an MONS in $M^\perp$, then $\{x_i\} \cup \{y_i\}$ is a MONS in V.

Proof. Suppose $z \perp x_i, y_j$ for all i and j. Then $z \perp M$, i.e., $z \in M^\perp$. The maximality of $\{y_j\}$ in $M^\perp$ implies $z = 0$. ■

Lemma 4. Under the condition of Lemma 3, $M^\perp \in \mathscr{L}_m(V)$.

Proof. Let $\{y_j\}$ and $\{z_j\}$ be two MONS in M. According to Lemma 3, $\{x_i\}\cup\{y_j\}$ and $\{x_i\}\cup\{z_j\}$ are two MONS in V. Therefore,

$$\sum_i m(\mathrm{sp}(x_i))+\sum_j m(\mathrm{sp}(y_j))=1=\sum_i m(\mathrm{sp}(x_i))+\sum_j m(\mathrm{sp}(z_j))$$

which gives $\sum_j m(\mathrm{sp}(y_j))=\sum_j m(\mathrm{sp}(z_j))$. ■

Lemma 5. Let v be a unit vector in the completion $\bar{V}$ of V. Then, for any $\varepsilon>0$, there exists a $\delta>0$ such that the following statement holds: If $w\in V$ is a unit vector such that $\|v-w\|<\delta$, then, for any finitely additive state m on $\mathscr{E}(V)$ and each $A\in\mathscr{E}(V)$ satisfying the properties $v\perp A$, $3\le\dim A<\infty$, we have the inequality

$$|m(A\vee\mathrm{sp}(w))-m(A)-m(\mathrm{sp}(w))|<\varepsilon \tag{5}$$

Proof. The proof is identical to that of Hamhalter and Pták (1987) and therefore is omitted. ■

Lemma 6. Let $\mathscr{S}_E(V)\neq\varnothing$. Under the conditions of Lemma 3,

$$\bigvee_i {}_L\, \mathrm{sp}(y_i)=M^\perp$$

Proof. First we show that if E is a finite-dimensional subspace of $M^\perp$, then

$$\bar{m}(M^\perp)\ge m(E) \tag{6}$$

for any $m\in\mathscr{S}_E(V)$. Indeed, let $\{e_k\}_{k=1}^t$ be an orthonormal basis in E. Choose an orthonormal system $\{z_s\}$ in $M^\perp$ such that $\{e_k\}_{k=1}^t\cup\{z_s\}$ is a MONS in M. By Lemma 4,

$$\bar{m}(M^\perp)=\sum_{k=1}^t m(\mathrm{sp}(e_k))+\sum_s m(\mathrm{sp}(z_s))\ge\sum_{k=1}^t m(\mathrm{sp}(e_k))=m(E)$$

Let $\{y_j\}$ be a MONS in $M^\perp$, and put

$$M_0=\bigvee_j {}_L\, \mathrm{sp}(y_j)$$

We assert that $M_0=M^\perp$. If not, then $\bar{M}_0\neq\overline{M^\perp}$, where the bar over M_0 and $M^\perp$ denotes the completion of M_0 and $M^\perp$ respectively. Hence, there is a $v\in\overline{M^\perp}$ that is orthogonal to $\bar{M}_0$.

According to Lemma 2, without loss of generality we may assume $m(\mathrm{sp}(z))>0$ for some unit vector $z\in M^\perp$. Put $\varepsilon=m(\mathrm{sp}(z))/2>0$. Applying Lemma 5 to $v\in\overline{M^\perp}$ and ε, we find a $w\in M^\perp$ with $\|w-v\|<\delta$ for some $\delta>0$ such that (5) holds for any finitely additive state.

Define a unitary operator $U: V \to V$ such that $Uz = w$ and $Ux = x$ for any $x \perp w, z$, and put $s(B) = m(U^{-1}(B))$ for any $B \in \mathscr{E}(V)$. Then $s \in \mathscr{S}_E(V)$. Since $\sum_i s(\mathrm{sp}(y_i)) < \infty$, there is at most a countable index set $J_0 = \{j_1, j_2, \ldots\}$ such that $s(\mathrm{sp}(y_j)) = 0$ for any $j \notin J_0$. Definite finite-dimensional subspaces $A_n = \mathrm{sp}(\{y_{j_1}, \ldots, y_{j_n}\})$. Then there is an integer n_0 such that

$$\bar{s}(M^\perp) = \sum_j s(\mathrm{sp}(y_j)) < s(A_{n_0}) + \varepsilon \tag{7}$$

Using the inequalities (5)–(7), we conclude

$$\begin{aligned} \bar{s}(M^\perp) &\geq s(A_{n_0} \vee \mathrm{sp}(w)) \\ &\geq s(A_{n_0}) + s(\mathrm{sp}(w)) - \varepsilon \\ &> \bar{s}(M^\perp) - \varepsilon + s(\mathrm{sp}(w) - \varepsilon = \bar{s}(M^\perp) \end{aligned}$$

which contradicts the beginning of the last inequality. ■

Lemma 7. Under the conditions of Lemma 6, $M \in \mathscr{L}_m(V)$ for any $m \in \mathscr{S}_E(V)$.

Proof. This follows immediately from Lemmas 6 and 4. ■

Lemma 8. (i) If $M \in \mathscr{L}_m(V)$ and if $\{x_i\}$ is a MONS in M, then

$$\bigvee_{i} {}_L \, \mathrm{sp}(x_i) = M$$

(ii) If $A, B \in \mathscr{L}_m(V)$, $A \subseteq B$, then $B = A \bigvee_L B \bigwedge_L A^\perp$, in particular, $M \in \mathscr{L}_m(V)$ implies $M^\perp \in \mathscr{L}_m(V)$.

(iii) If $\{M_t: t \in T\}$ is a system of mutually orthogonal subspaces from $\mathscr{L}_m(V)$, then

$$\bigvee_{t \in T} {}_L \, M_t \in \mathscr{L}_m(V)$$

(iv) $\bar{m}|\mathscr{L}_m(V)$ is a completely additive state on a quantum logic $\mathscr{L}_m(V)$.

Proof. (i) This follows the same idea as the proof of Lemma 6.

(ii) Let $A, B \in \mathscr{L}_m(V)$, $A \subseteq B$. Choose a MONS $\{a_i\}$ in A and an orthonormal system $\{b_j\}$ in B such that $\{a_i\} \cup \{b_j\}$ is a MONS in B. In view of (i),

$$A = \bigvee_{i} {}_L \, \mathrm{sp}(a_i), \qquad B = \bigvee_{i} {}_L \, \mathrm{sp}(a_i) \underset{L}{\vee} \bigvee_{j} {}_L \, \mathrm{sp}(b_j)$$

we put

$$B_0 = \bigvee_{j} {}_L \, \mathrm{sp}(b_j)$$

then $B_0 \subseteq B \cap A^{\perp}$. On the other hand,

$$B = A \bigvee_L B_0 \subseteq A \underset{L}{\vee} B \cap A^{\perp} \subseteq B$$

(iii) For any $t \in T$, let $\{x_t^s : s \in T_t\}$ be a MONS in M_t. Then

$$M := \bigvee_{t \in T} {}_L M_t = \bigvee_{t \in T} {}_L \bigvee_{s \in T_t} {}_L \operatorname{sp}(x_t^s)$$

so that, according to Lemma 7, $M \in \mathscr{L}_m(V)$.

(iv) It is now evident that $\mathscr{L}_m(V)$ is a quantum logic in the sense of Varadarajan (1962) and $\bar{m} | \mathscr{L}_m(V)$ is a completely additive state on $\mathscr{L}_m(V)$. ■

Lemma 9. Let $\{x_i\}_{i=1}^{\infty}$ be any sequence of orthonormal vectors from V. Then

$$N \subseteq M := \bigvee_{i=1}^{\infty} {}_L \operatorname{sp}(x_i), \qquad N \in \mathscr{L}(V)$$

is an element of $\mathscr{L}_m(V)$.

Proof. Choose in N two MONS $\{a_i\}$ and $\{b_i\}$ and define

$$A = \bigvee_i {}_L \operatorname{sp}(a_i), \qquad B = \bigvee_i {}_L \operatorname{sp}(b_i)$$

Applying the Gram-Schmidt orthogonalization process to $a_1, b_1, a_2, b_2, \ldots$, we find orthonormal vectors $\{c_n\}_{n=1}^{\infty}$ such that

$$\bigvee_{i=1}^{2n} \operatorname{sp}(c_i) = \bigvee_{i=1}^{n} \operatorname{sp}(a_i) \vee \bigvee_{i=1}^{n} \operatorname{sp}(b_i)$$

Let

$$C = \bigvee_{i=1}^{\infty} {}_L \operatorname{sp}(c_i) \subseteq N$$

Then $A, B, C \in \mathscr{L}_m(V)$ and, due to the maximality of $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ in N, we see that they are also maximal in C. This implies $A = B = C$ and $\sum_i m(\operatorname{sp}(a_i)) = \sum_i m(\operatorname{sp}(b_i))$. Consequently, N definitely belongs to $\mathscr{L}_m(V)$. ■

Now we are ready to prove the main result of the paper; it is of interest to present two different proofs.

Proof 1. We show that, for any sequence of orthonormal vectors $\{x_i\}_{i=1}^{\infty}$ from V, the space

$$M = \bigvee_{i=1}^{\infty} {}_L \operatorname{sp}(x_i)$$

is splitting. Indeed, let us write $\mathscr{L}(0, M) = \{N: N \in \mathscr{L}(V), N \subseteq M\}$. In view of Lemmas 8 and 9, $\mathscr{L}(0, M) \subseteq \mathscr{L}(V)$ and it is a quantum logic in the sense of Varadarajan (1962) with respect to a relative orthocomplement $'$ defined via

$$N' := M \cap N^{\perp} \in \mathscr{L}(0, M)$$

It is evident that if we define $\mathscr{L}(M) = \{N \subseteq M: (N^{\perp M})^{\perp M} = N\}$, where $N^{\perp M} = \{x \in M: (x, y) = 0 \text{ for all } y \in N\}$, then $N^{\perp} \cap M = N^{\perp M}$.

Let $N \in \mathscr{L}(0, M)$. Then $N^{\perp} \cap M = N' = N^{\perp M}$, so that $N'' = N$ and $(N^{\perp M})^{\perp M} = N$, which gives $N \in \mathscr{L}(M)$. Conversely, let $N \in \mathscr{L}(M)$; we show that $N \in \mathscr{L}(V)$. Indeed, since $N \subseteq N^{\perp\perp}$, we conclude that if x is an arbitrary vector from V orthogonal to $N^{\perp}$, then $x \perp N^{\perp} \supseteq M^{\perp}$, so that $x \in M^{\perp\perp} = M$. It is clear that for any subset $A \subseteq M$, $A^{\perp} \supseteq A^{\perp M}$. Hence,

$$N = N \cap M \subseteq N^{\perp\perp} = N^{\perp\perp} \cap M = (N^{\perp})^{\perp M} \subseteq (N^{\perp M})^{\perp M} = N$$

and $N \in \mathscr{L}(V)$; consequently, $N \in \mathscr{L}(0, M)$.

The equality $\mathscr{L}(0, M) = \mathscr{L}(M)$ implies, in view of Lemma 8, that $\mathscr{L}(M)$ is orthomodular. The result of Amemiya and Araki (1966) asserts that in this case M is complete, and we know that any complete subspace of V is splitting.

The criterion of Dvurečenskij (1988b) says that V is complete iff $\mathscr{L}(V)$ contains the join of any sequence of orthogonal one-dimensional subspaces of V, so that V is complete.

Proof. 2. We show that if

$$A = \bigvee_{i} {}_L \operatorname{sp}(x_i)$$

where $\{x_i\}$ is any nonvoid system of orthonormal vectors from V, then A is splitting. Let $y \in V$, $y \notin A$. From the covering property it follows that there exists a $y_1 \in V$ such that $\{x_i\} \cup \{y_1\}$ is a MONS in $A \vee_L \operatorname{sp}(y)$. As in the proof of Lemma 6, we may find a completely additive state s on $\mathscr{E}(V)$ such that

$$\sum_i m(\operatorname{sp}(x_i)) < \sum_i s(\operatorname{sp}(x_i)) + s(\operatorname{sp}(y_1))$$

which gives $y_1 \neq 0$. In any inner product space we have

$$M \vee_L C = M + C$$

whenever $M \in \mathscr{L}(V)$ and C is finite-dimensional. Using the covering property, $A \vee_L \operatorname{sp}(y) = A \vee_L \operatorname{sp}(y_1) = A + \operatorname{sp}(y_1)$. This shows that A is splitting.

Now let $M \in \mathscr{L}(V)$ be given and choose a MONS $\{y_i\}$ in M. It is clear that

$$A := \bigvee_{i} {}_L \operatorname{sp}(y_i) \subseteq M$$

On the other hand, let $x \in M$. Since A is splitting, $x = x_A + x_{A^\perp}$, where $x_A \in A$, $x_{A^\perp} \in A^\perp$. The maximality of $\{y_i\}$ in M entails $x_{A^\perp} = 0$. Therefore, $\mathscr{L}(V) = \mathscr{E}(V)$, i.e., V is complete.

The theorem is completely proved. ■

In conclusion, we note that $\mathscr{E}(V)$, for V incomplete, gives an example of an orthocomplemented, orthomodular poset with a quite full system of finitely additive states that possesses no completely additive state. In particular, the example of Gudder (1974) described above gives a stateless $\mathscr{E}(V)$.

Finally, we note that Dvurečenskij and Mišík (1988) proved that any state on $\mathscr{L}(V)$ of an inner product space V whose dimension is a nonmeasurable cardinal is completely additive. For the state on $\mathscr{E}(V)$ this conclusion is unknown to us.

REFERENCES

Amemiya, I., and Araki, H. (1966). A remark on Piron's paper, *Publication of the Research Institute for Mathematical Sciences, Series A*, **2**, 423–427.

Cattaneo, G., and Marino, G. (1986). Completeness of inner product spaces with respect to splitting subspaces. *Letters in Mathematical Physics*, **11**, 15–20.

Dvurečenskij, A. (1988a). Note on a construction of unbounded measures on a nonseparable Hilbert space quantum logic. *Annales de l'Institut Henri Poincaré-Physique Theorique*, **48**, 297–310.

Dvurečenskij, A. (1988b). Completeness of inner product spaces and quantum logic of splitting subspaces. *Letters in Mathematical Physics*, **15**, 231–235.

Dvurečenskij, A. (1989). A state criterion of the completeness for inner product spaces, *Demonstratio Mathematique*, in press.

Dvurečenskij, A., and Mišík, Jr., L. (1988). Gleason's theorem and completeness of inner product spaces, *International Journal of Theoretical Physics*, **27**, 417–426.

Gross, H., and Keller, H. A. (1977). On the definition of Hilbert space. *Manuscripta Mathematica*, **23**, 67–90.

Gudder, S. P. (1974). Inner product spaces, *American Mathematical Monthly*, **81**, 29–36.

Hamhalter, J., and Pták, P. (1987). A completeness criterion for inner product spaces, **19**, 259–263.

Maeda, S. (1980). *Lattice Theory and Quantum Logic*, Mahishoten, Tokyo (in Japanese).

Varadarajan, V. S. (1962). Probability in physics and a theorm on simultaneous observability, *Communications in Pure and Applied Mathematics*, **15**, 186–217 [Errata, **18** (1965)].

Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. 1, Van Nostrand, Princeton, New Jersey.